Walrasian Equilibrium

Disaggregated treatment of individual behavior,
and behavior that need not be single-valued

As we've noted, our approach to studying equilibrium so far has depended upon individuals' demand functions being single-valued and defined for all price-lists. This dependence was highlighted by our study of the existence-of-equilibrium problem, which relied on the market demand function and failed to cover such non-pathological cases as Cobb-Douglas preferences or linear preferences.

The definition of market equilibrium we're about to give is still that all markets clear. But by also including the individuals' choices as an explicit element of the equilibrium, we can deal with situations in which some price-lists leave some individuals indifferent among many available choices; and situations in which some price-lists leave some individuals with no optimal choice at all. But most important, framing things explicitly in terms of individual behavior will help us understand a much broader range of phenomena.

Definition: Let $E = \big((u^i, \mathring{\mathbf{x}}^i)\big)_1^n$ be an economy made up of n consumers $(u^1, \mathring{\mathbf{x}}^1)$, $\ldots$, $(u^n, \mathring{\mathbf{x}}^n)$. A *Walrasian equilibrium* of E is a pair $(\mathbf{p}^*, \mathbf{x}^*) \in \mathbb{R}^l_+ \times \mathbb{R}^{nl}_+$ that satisfies

(1) $\forall i \in \{1, \ldots, n\}$: $\mathbf{x}^{*i}$ maximizes u^i on the budget set

$$B(\mathbf{p}^*, \mathring{\mathbf{x}}^i) := \{\, \mathbf{x}^i \in \mathbb{R}^l_+ \mid \mathbf{p}^* \cdot \mathbf{x}^i \leqq \mathbf{p}^* \cdot \mathring{\mathbf{x}}^i \,\},$$

and

(2) $\forall k \in \{1, \ldots, l\}$: $\sum_{i=1}^n (x_k^{*i} - \mathring{x}_k^i) \leqq 0$ and $\sum_{i=1}^n (x_k^{*i} - \mathring{x}_k^i) = 0$ if $p_k^* > 0$.

Note that in this definition an equilibrium consists of both a price-list $\mathbf{p}^*$ *and* an allocation $\mathbf{x}^* = (\mathbf{x}^{*i})_1^n = (\mathbf{x}^{*1}, \ldots, \mathbf{x}^{*n})$ that specifies the bundle each consumer receives.

Condition (1) says that at $\mathbf{p}^*$ the bundles $\mathbf{x}^{*1}, \ldots, \mathbf{x}^{*n}$ are consistent with the individual consumers' demand correspondences: $\mathbf{x}^{*i} \in D^i(\mathbf{p}^*)$ for each i.

Condition (2) says that all markets clear: no good is in excess demand, and any good in excess supply has a zero price.

Existence of Walrasian Equilibrium

This theorem allows for undefined demands (as would happen, for example, if a price were zero and some utility function were strictly increasing) and for weak convexity of preferences.

Theorem: Let $E = ((u^i, \mathring{\mathbf{x}}^i))_1^n$ be an exchange economy. If each consumer $(u^i, \mathring{\mathbf{x}}^i)$ satisfies the conditions

(a) u^i is continuous, increasing, and quasi-concave, and

(b) $\mathring{x}_k^i > 0, \quad k = 1, \ldots, l,$

then E has a Walrasian equilibrium.

Proof:

For each i, let $D^i \colon S \twoheadrightarrow \mathbb{R}_+^l$ and $\zeta^i \colon S \twoheadrightarrow \mathbb{R}^l$ be consumer i's demand and net demand correspondences, restricted to the simplex S in $\mathbb{R}^l$:

$$\begin{aligned} D^i(\mathbf{p}) &:= \{\mathbf{x}^i \in \mathbb{R}_+^l \mid \mathbf{x}^i \text{ maximizes } u^i \text{ on } B(\mathbf{p}, \mathring{\mathbf{x}}^i\}, \text{ and} \\ \zeta^i(\mathbf{p}) &:= D^i(\mathbf{p}) - \mathring{\mathbf{x}}^i. \end{aligned}$$

We define a "truncated version" of the economy E, in which the consumers' chosen bundles are constrained to lie within a given compact set K. To begin with, let $\beta := 1 + \max\{\mathring{x}_k \mid k = 1, \ldots, l\}$, where $\mathring{\mathbf{x}} = \sum_{i=1}^n \mathring{\mathbf{x}}^i$, and let K denote the cube

$$K := \{\mathbf{x} \in \mathbb{R}_+^l \mid |x_k| \leqq \beta, k = 1, \ldots, l\}.$$

For each i, define $\varphi^i \colon S \twoheadrightarrow \mathbb{R}_+^l$, $\widehat{D}^i \colon S \twoheadrightarrow \mathbb{R}_+^l$, and $\widehat{\zeta}^i \colon S \twoheadrightarrow \mathbb{R}^l$ by

$$\begin{aligned} \varphi^i(\mathbf{p}) &:= B(\mathbf{p}, \mathring{\mathbf{x}}^i) \cap K, \\ \widehat{D}^i(\mathbf{p}) &:= \{\mathbf{x}^i \in \mathbb{R}_+^l \mid \mathbf{x}^i \text{ maximizes } u^i \text{ on } \varphi^i(\mathbf{p})\} \quad \text{, and} \\ \widehat{\zeta}^i(\mathbf{p}) &:= \widehat{D}^i(\mathbf{p}) - \mathring{\mathbf{x}}^i. \end{aligned}$$

Note that $\varphi^i(\mathbf{p})$ is the consumer's truncated budget set, $\widehat{D}^i(\cdot)$ is his truncated demand correspondence, and $\widehat{\zeta}^i(\cdot)$ is his truncated net demand correspondence. The set K is compact, each u^i is continuous, and each φ^i is a continuous correspondence; therefore the Maximum Theorem guarantees that each $\widehat{D}^i$ has a closed graph. Therefore each $\widehat{\zeta}^i$ has a closed graph as well, and consequently the truncated economy's excess demand correspondence $\widehat{\zeta}(\cdot) := \sum_1^n \widehat{\zeta}^i(\cdot)$ also has a closed graph.

We will first show that $\widehat{\zeta}$ has a "market level" (Arrow & Hahn-type) equilibrium – i.e., that

(1) there is a $\mathbf{p}^* \in S$ and a $\mathbf{z}^* \in \mathbb{R}^l$ for which $\mathbf{z}^* \in \widehat{\zeta}^i(\mathbf{p}^*)$ and $\mathbf{z}^* \leqq \mathbf{0}$ and $p_k^* > 0 \Rightarrow z_k^* = 0$.

Then we will

(2) find an allocation $(\mathbf{x}^{*i})_1^n$ that is consistent with the market net demand $\mathbf{z}^*$.

Finally we will show that

(3) $(\mathbf{p}^*, (\mathbf{x}^{*i})_1^n)$ is a Walrasian equilibrium for E.

We use a fixed-point argument to establish (1). Note that because K is compact and each u^i is continuous, each $\widehat{\zeta}^i$ must be non-empty-valued on S and therefore $\widehat{\zeta}$ is also non-empty-valued on S. If $\widehat{\zeta}$ is also single-valued, we can simply apply our existence result from Arrow & Hahn. More generally (since $\widehat{\zeta}$ may be multi-valued), we define a "price adjustment" correspondence $\mu\colon K \twoheadrightarrow S$ as follows:

$$\mu(\mathbf{z}) := \{\mathbf{p} \in S \mid \mathbf{p} \text{ maximizes } \mathbf{p} \cdot \mathbf{z} \text{ on } S\}$$

and a "transition" correspondence $f\colon S \times K \twoheadrightarrow S \times K$ by

$$f(\mathbf{p}, \mathbf{z}) := \mu(\mathbf{z}) \times \widehat{\zeta}(\mathbf{p}).$$

Note that in this construction we take the state space to be $S \times K$, not just S, so that a state is both a price-list *and* a net demand bundle; and that just as in our previous existence proof, the transition function/correspondence f is "made up" – it does not represent the actual dynamic path of prices and quantities. As in the Arrow & Hahn proof, we will show that f has a fixed point $(\mathbf{p}^*, \mathbf{z}^*)$ and then show that any such $(\mathbf{p}^*, \mathbf{z}^*)$ must satisfy (1).

We know that $\widehat{\zeta}$ has a closed graph and is non-empty-valued and convex-valued, and it is easy to show that μ has the same properties. Therefore so does f, and Kakutani's Theorem therefore implies that f has a fixed point.

Let $(\mathbf{p}^*, \mathbf{z}^*)$ be a fixed point of f. Because $\mathbf{p}^* \in \mu(\mathbf{z}^*)$, we have $\mathbf{p} \cdot \mathbf{z}^* \leqq \mathbf{p}^* \cdot \mathbf{z}^*$ for every $\mathbf{p} \in S$. We also have $\mathbf{p}^* \cdot \mathbf{z}^* = 0$, because $\mathbf{z}^* \in \widehat{\zeta}(\mathbf{p}^*)$, and because $\widehat{\zeta}$ satisfies Walras's

Law (each u^i is strictly increasing). Combining $\mathbf{p}\cdot\mathbf{z}^* \leqq \mathbf{p}^*\cdot\mathbf{z}^*$ (for all $\mathbf{p}\in S$) and $\mathbf{p}^*\cdot\mathbf{z}^* = 0$ yields $\mathbf{p}\cdot\mathbf{z}^* \leqq 0$ for every $\mathbf{p}\in S$, from which it follows that $\mathbf{z}^* \leqq \mathbf{0}$ – i.e., that $z_k^* \leqq 0$ for each k. (Suppose, for example, that $z_1^* > 0$. Then $(1,0,\ldots,0)\cdot\mathbf{z}^* > 0$. But we have shown that $\mathbf{p}\cdot\mathbf{z}^* \leqq 0$ for every $\mathbf{p}\in S$.) The final part of (1) follows from our familiar argument: since each $p_k^* \geqq 0$ and each $z_k^* \leqq 0$, and $\mathbf{p}^*\cdot\mathbf{z}^* = 0$, we have $p_k^* > 0 \Rightarrow z_k^* = 0$.

In order to accomplish step (2), we use the $\mathbf{z}^*$ from (1), as follows: Since $\mathbf{z}^* \in \widehat{\zeta}(\mathbf{p}^*) = \sum_1^n \widehat{\zeta}^i(\mathbf{p}^*)$, there are bundles $\mathbf{z}^{*1},\ldots,\mathbf{z}^{*n}$ such that $\mathbf{z}^{*i} \in \widehat{\zeta}^i(\mathbf{p}^*)$ for each i and such that $\sum_1^n \mathbf{z}^{*i} = \mathbf{z}^*$. But $\widehat{\zeta}^i(\mathbf{p}^*) = \widehat{D}^i(\mathbf{p}) - \mathring{\mathbf{x}}^i$. Thus, for any $\mathbf{z}^{*i} \in \widehat{\zeta}^i(\mathbf{p}^*)$, there is an $\mathbf{x}^{*i} \in \widehat{D}^i(\mathbf{p}^*)$ such that $\mathbf{z}^{*i} = \mathbf{x}^{*i} - \mathring{\mathbf{x}}^i$ – namely $\mathbf{x}^{*i} = \mathring{\mathbf{x}}^i + \mathbf{z}^{*i}$. This gives us the allocation $(\mathbf{x}^{*i})_1^n$ in (2).

We now establish (3) – that $(\mathbf{p}^*, (\mathbf{x}^{*i})_1^n)$ is a Walrasian equilibrium for E. The market-clearing equilibrium condition, $\sum_1^n \mathbf{x}^{*i} \leqq \sum_1^n \mathring{\mathbf{x}}^i$, is straightforward: $\mathbf{z}^* = \sum_1^n \mathbf{z}^{*i} = \sum_1^n (\mathbf{x}^{*i} - \mathring{\mathbf{x}}^i)$, from (2), and $\mathbf{z}^* \leqq \mathbf{0}$, from (1). It remains only to establish the utility-maximizing equilibrium condition, that each $\mathbf{x}^{*i}$ is in $D^i(\mathbf{p}^*)$, not merely in $\widehat{D}^i(\mathbf{p}^*)$ – in other words, that at $\mathbf{p}^*$ the truncation of each consumer's budget set to the cube K is not actually binding, i.e., that at $\mathbf{p}^*$ the consumer would choose $\mathbf{x}^{*i}$ even if all of $B(\mathbf{p}^*, \mathring{\mathbf{x}}^i\}$ were available. We have $\mathbf{x}^{*i} \in \widehat{D}^i(\mathbf{p}^*)$ for each i, from (2). Moreover, we also have $x_k^{*i} \leqq \sum_1^n x_k^{*j} \leqq \sum_1^n \mathring{x}_k^j < \beta$ for each k; therefore each $\mathbf{x}^{*i}$ is in the *interior* of K. Now suppose that $\mathbf{x}^{*i} \notin D^i(\mathbf{p}^*)$ for some i – i.e., there is an $\tilde{\mathbf{x}}^i \in \mathbb{R}$ such that $\mathbf{p}^*\cdot\tilde{\mathbf{x}}^i \leqq \mathbf{p}^*\cdot\mathring{\mathbf{x}}^i$ and $u^i(\tilde{\mathbf{x}}^i) > u^i(\mathbf{x}^{*i})$. Since the budget set $B(\mathbf{p}^*, \mathring{\mathbf{x}}^i\}$ is convex, every bundle on the line segment $[\tilde{\mathbf{x}}^i, \mathbf{x}^{*i}]$ is also in $B(\mathbf{p}^*, \mathring{\mathbf{x}}^i\}$; and since $u^i(\tilde{\mathbf{x}}^i) > u^i(\mathbf{x}^{*i})$ and u^i is quasi-concave, every bundle $\mathbf{x}^i$ on this line segment also satisfies $u^i(\mathbf{x}^i) > u^i(\mathbf{x}^{*i})$. Further, every neighborhood of $\mathbf{x}^{*i}$ contains bundles on this line segment; and since $\mathbf{x}^{*i} \in intK$, there are bundles $\mathbf{x}^i \in B(\mathbf{p}^*, \mathring{\mathbf{x}}^i\} \cap K$ that lie on the line segment and which therefore satisfy $u^i(\mathbf{x}^i) > u^i(\mathbf{x}^{*i})$. Consequently, $\mathbf{x}^{*i} \notin \widehat{D}^i(\mathbf{p}^*)$, contrary to what we have already shown, and therefore we must have $\mathbf{x}^{*i} \in D^i(\mathbf{p}^*)$ after all. $\|$

Figure 1 depicts the argument in the final paragraph of the proof. Figure 2 shows why the argument requires that $\mathbf{x}^{*i} \in intK$.

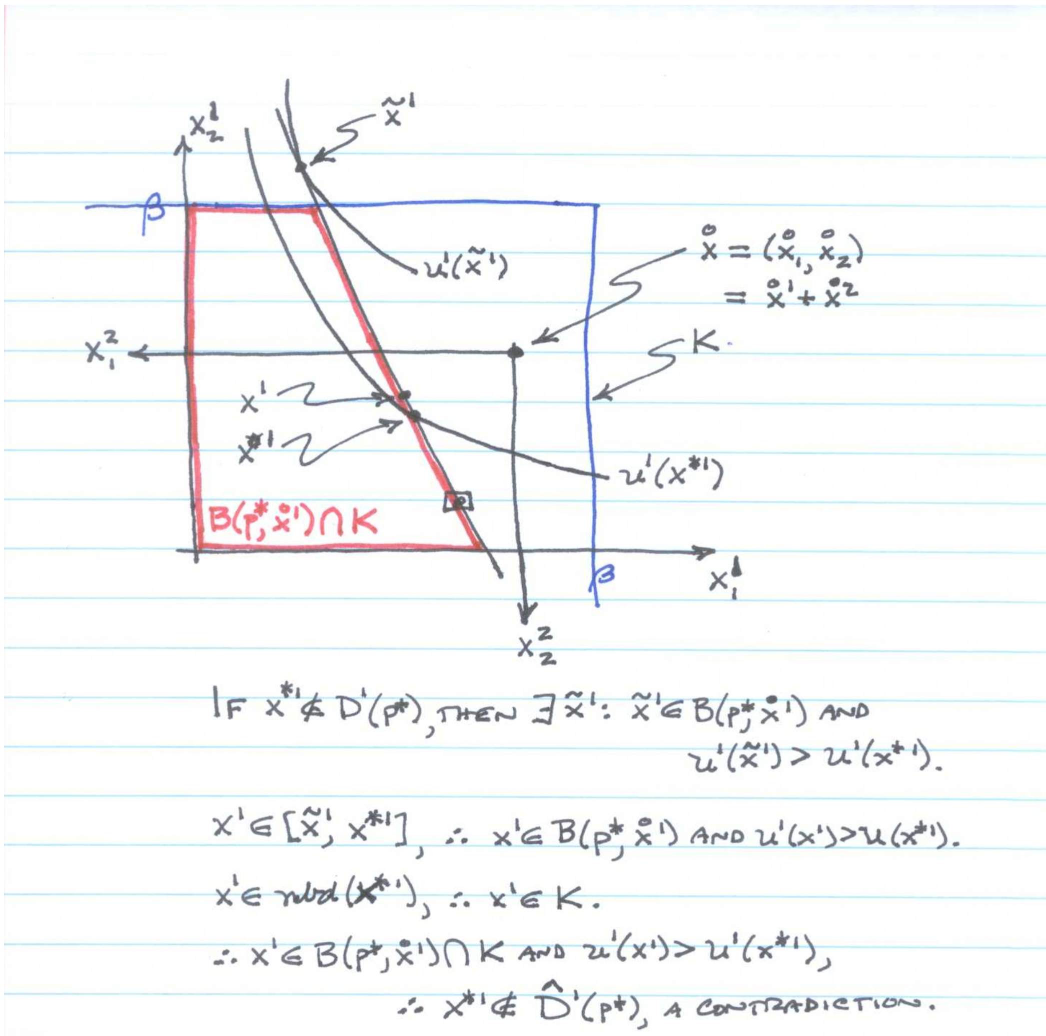

Figure 1: The argument showing that $\mathbf{x}^{*i} \in D^i(\mathbf{p}^*)$

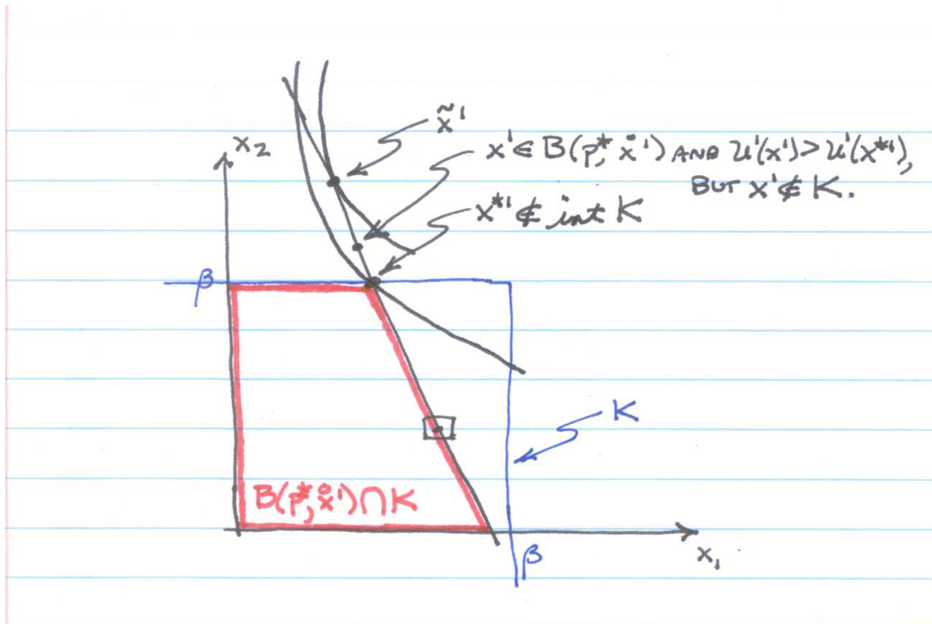

Figure 2: The argument doesn't work if $\mathbf{x}^{*i} \notin intK$